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# Bound States of a Gravitational Soliton and a Test Particle

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Classical and quantum bound states of a test particle in the regular gravitational field of a gravitational soliton are investigated. The quantum spectrum is very similar to that of a Newtonian atom, except for the absence of $s$ orbitals.

## 1. INTRODUCTION

Gravitational solitons are regular static solutions of Einstein–Higgs (EH) (Kodama, 1978), Einstein–Maxwell–Higgs (EMH) (Bronnikov, 1973; Clément, 1981a), or Einstein–Yang–Mills–Higgs (EYMH) (Clément, 1981b) equations, in the case of a repulsive Higgs field (coupled negatively to gravity). These solutions have in common a spatial geometry with two symmetrical asymptotically flat regions.

In two preceding papers (Chetouani and Clément, 1984; Clément, 1984) we have studied the classical and quantum scattering of test particles by a light gravitational soliton (in which case Newtonian effects may be neglected before spatial geometrical effects). We now complete this study by the investigation of bound states of test particles in the gravitational field of the soliton. Because this field is regular, a discrete number of bound states survive quantization, contrary to the case of the gravitational field of a point particle [in which test particles tunnel through the centrifugal barrier into the Schwarzschild black hole (Misner et al., 1973)]. Thus we can have gravitational atoms.

For simplicity, our study is restricted to the case of a spinless, neutral test particle in the field of an EMH or EYMH soliton, in the case of a



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vanishing self-coupling of the Higgs field [in this case the EH soliton is massless; bound states of a Dirac particle with a massive EH soliton have been studied numerically by de Oliveira (1982)]. The classical bound state spectrum is derived in Section 2 of this paper. Section 3 is devoted to the study of quantum bound states, for which we approximately determine the lowest energy levels for each value of the orbital quantum number $l$.

## 2. CLASSICAL BOUND STATES

The geometry of the EMH or EYMH soliton is given (in the case of a vanishing self-coupling of the Higgs field) by

$$ds^2 = g_{00}\,dt^2 - g_{00}^{-1}\left[du^2 + (\rho_0^2 + u^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \tag{1}$$

with

$$g_{00} = \frac{\cos^2(\pi\lambda/2)}{\cos^2\lambda\eta}$$

$$\eta = \arctan\left(\frac{u}{\rho_0}\right) \tag{2}$$

The dimensionless constant $\lambda$ and characteristic length $\rho_0$ are related to the mass $M$ and electric or magnetic charge $Q$ of the soliton by

$$\sin(\pi\lambda/2) = (4\pi G)^{1/2}\frac{M}{Q}$$

$$\rho_0 = \frac{Q^2}{8M}\,\frac{\sin(\pi\lambda/2)}{\pi\lambda/2} \tag{3}$$

where $G$ is Newton's constant.

The first-order differential equations for time- or lightlike geodesics in a spherically symmetric static metric are (in the plane $\theta = \pi/2$)

$$-g_{00}^{-1}g_{\varphi\varphi}\frac{d\varphi}{dt} = \sigma$$

$$-g_{00}^{-2}g_{rr}\left(\frac{dr}{dt}\right)^2 - g_{\varphi\varphi}^{-1}\sigma^2 - g_{00}^{-1} = -1 + \beta^2 \tag{4}$$

where $\sigma$ and $\beta^2(\beta^2 \leqslant 1$, may be negative) are integration constants. We assume $\lambda \ll 1$, so that the gravitational potential is small, and we may approximate

$$g_{00} \simeq 1 + \lambda^2\left(\eta^2 - \frac{\pi^2}{4}\right) \tag{5}$$

The equation for the radial motion of a test particle is then

$$\left(\frac{du}{dt}\right)^2 + U(u) = 0 \tag{6}$$

with

$$U = \frac{\sigma^2}{\rho_0^2}\cos^2\eta + \lambda^2\left(\eta^2 - \frac{\pi^2}{4}\right) - \beta^2 \tag{7}$$

$\eta$ being related to $u$ by (2).

For $\beta^2 \geqslant 0$, the particle can go to infinity, so that there are no bound states, unless the effective potential $U(u)$ has at least two maxima. It can be checked that this is not the case.

For those values of $\beta^2 < 0$ such that $U_{\min} \leqslant 0$, the particle is bound. The extrema of $U(\eta)$ being solutions of

$$2\lambda^2\eta = \frac{\sigma^2}{\rho_0^2}\sin 2\eta \tag{8}$$

we must distinguish between two cases:

(a) $\sigma \leqslant \lambda\rho_0$. The only solution of equation (8) is $\eta = 0$, i.e., $u = 0$, which gives a minimum of $U$, with the value

$$U_{\min} = \frac{\sigma^2}{\rho_0^2} - \lambda^2\frac{\pi^2}{4} - \beta^2 \tag{9}$$

The bound state spectrum then corresponds to

$$\frac{\sigma^2}{\rho_0^2} - \lambda^2\frac{\pi^2}{4} \leqslant \beta^2 < 0 \tag{10}$$

(b) $\sigma > \lambda\rho_0$. Then equation (8) has the three solutions $\eta = 0$, which corresponds to a local maximum, and $\eta = \pm\,\eta_0$, which correspond to degenerate minima, because

$$\frac{d^2U}{d\eta^2}(\eta_0) = -\frac{2\sigma^2}{\rho_0^2}\cos 2\eta_0 + 2\lambda^2 > -\frac{2\sigma^2}{\rho_0^2}\frac{\sin 2\eta_0}{2\eta_0} + 2\lambda^2 = 0 \tag{11}$$

In this case the bound state spectrum is given by

$$\lambda^2\left(\frac{\eta_0}{\tan\eta_0}+\eta_0^2-\frac{\pi^2}{4}\right)\leqslant\beta^2<0 \tag{12}$$

[the left-hand side of (12) can be shown to be strictly negative].

## 3. QUANTUM BOUND STATES

The stationary Klein–Gordon equation for a scalar particle of mass $m$ is

$$\hbar^2|g|^{-1/2}\partial_i\left[|g|^{1/2}g^{ij}\partial_j\psi\right]+\left(m^2-E^2g_{00}^{-1}\right)\psi=0 \tag{13}$$

Putting

$$g_{ij}=g_{00}^{-1}\tilde{g}_{ij} \tag{14}$$

where $\tilde{g}_{ij}$ is the Ellis metric, we rewrite equation (13) as

$$\hbar^2|\tilde{g}|^{-1/2}\partial_i\left[|\tilde{g}|^{1/2}\tilde{g}^{ij}\partial_j\psi\right]+g_{00}^{-1}\left(m^2-E^2g_{00}^{-1}\right)\psi=0 \tag{15}$$

and use the results of Clément (1984) for the separation of this equation in spherical coordinates. The expansion of $\psi$ in spherical harmonics

$$\psi(\mathbf{x})=\sum_{l=0}^{\infty}\left(\rho_0^2+u^2\right)^{-1/2}f_l(u)Y_l^m(\theta,\varphi) \tag{16}$$

reduces (15) to the radial equation

$$-\hbar^2f_l''(u)+\left[\frac{\hbar^2\rho_0^2}{\left(\rho_0^2+u^2\right)^2}+\frac{\hbar^2l(l+1)}{\rho_0^2+u^2}+g_{00}^{-1}\left(m^2-E^2g_{00}^{-1}\right)\right]f_l(u)=0 \tag{17}$$

which we may write, approximating again $g_{00}$ by (5), as

$$-\hbar^2f''(u)+V(u)f(u)=0 \tag{18}$$

with

$$V=\frac{\hbar^2}{\rho_0^2}\left[\cos^4\eta+l(l+1)\cos^2\eta+\alpha\left(\eta^2-\frac{\pi^2}{4}\right)\right]+m^2-E^2 \tag{19}$$

where

$$\alpha = (2E^2 - m^2)\frac{\lambda^2\rho_0^2}{\hbar^2} \tag{20}$$

Following further Clément (1984), we insist that the states of a neutral particle be eigenstates of the charge conjugation operator, defined by the discrete transformation [the "new" symmetry of metric (1)]

$$u \to -u \tag{21}$$

This means that the radial wave function $f_l(u)$ must either be even or odd under this transformation. The additional requirement that we recover usual flat-space quantum mechanics in the limit $\rho_0 \to 0$ then selects odd wave functions

$$f_l(-u) = -f_l(u) \tag{22}$$

which of course are also regular wave functions. Because of this condition, we may now assume $u \geqslant 0$.

The quantum effective potential $V(\eta)$ is proportional to the classical potential $U(\eta)$ given by (7) (the correspondence is $\sigma = \hbar^2 l(l+1)/E^2$, $1-\beta^2 = m^2/E^2$), with additional quantum corrections. However, the behaviors of these functions are qualitatively the same. We will concentrate in the following discussion on the approximate determination of the lowest energy level for a given $l$.

The extrema of the potential $V(\eta)$ are the solutions of

$$2\alpha\eta = \left[l(l+1)+2\cos^2\eta\right]\sin 2\eta \tag{23}$$

Here again we distinguish between two cases:

(a) $l(l+1) \leqslant \alpha - 2$. The only solution of equation (23) is $\eta = 0$, which gives a minimum.

(b) $l(l+1) > \alpha - 2$. $\eta = 0$ is now a maximum, but equation (23) has, for $\alpha > 0$, another solution $\eta = \eta_0$, which is a minimum because

$$\begin{aligned}\frac{d^2V}{d\eta^2}(\eta_0) &= \frac{2\hbar^2}{\rho_0^2}\left[6\cos^2\eta_0\sin^2\eta_0 - 2\cos^4\eta_0 - l(l+1)\cos 2\eta_0 + \alpha\right] \\ &> \frac{8\hbar^2}{\rho_0^2}\cos^2\eta_0\sin^2\eta_0 > 0\end{aligned} \tag{24}$$

The parameter $\alpha$ being an increasing function of $E$, the lowest energy level $E_0$ for a given $l$ belongs to case (b), and lies above the value given by

$V(\eta_0) = 0$ because of the uncertainty relations. A standard quantum mechanical argument applied to the one-dimensional Schrödinger equation (18) leads to

$$0 = \langle p_u^2 + V \rangle \simeq V(u_0) + \langle p_u^2 + \tfrac{1}{2} V''(u_0)(u - u_0)^2 \rangle$$
$$\gtrsim V(u_0) + \hbar \left[ \tfrac{1}{2} V''(u_0) \right]^{1/2} \tag{25}$$

Let us discuss this inequality separately for the cases $l = 0$ (in which case spherical symmetry actually leads to a stronger inequality, which we shall not need) and $l \neq 0$.

For $l = 0$, taking into account the inequality (24) and the relation

$$\frac{d^2 V}{du^2}(u_0) = \frac{d^2 V}{d\eta^2}(\eta_0) \cdot \frac{1}{p_0^2} \cos^4 \eta_0 \tag{26}$$

we derive from (25) the inequality

$$\frac{\hbar^2}{p_0^2} \left[ \cos^4 \eta_0 + 2 \cos^3 \eta_0 \sin \eta_0 + \alpha \left( \eta_0^2 - \frac{\pi^2}{4} \right) \right] \lesssim E^2 - m^2 \tag{27}$$

Now we note that for the bound states to be stable (no tunneling) we must have $E^2 < m^2$, hence

$$\cos^4 \eta_0 + 2 \cos^3 \eta_0 \sin \eta_0 \lesssim \alpha \left( \frac{\pi^2}{4} - \eta_0^2 \right) \tag{28}$$

or, using (23),

$$1 + \frac{\operatorname{cotan} \eta_0}{2} \lesssim \frac{\pi^2}{4 \eta_0} - \eta_0 \tag{29}$$

which gives

$$\eta_0 \lesssim 1.05 \tag{30}$$

Using again the condition $E^2 < m^2$, the definition (20) of $\alpha$, and its relation (23) to $\eta_0$, it follows that

$$\frac{m^2 \lambda^2 p_0^2}{\hbar^2} > \alpha \gtrsim 0.147 \tag{31}$$

or (inserting the appropriate power of $c$)

$$m \gtrsim m_0 = 1.36\left(\frac{\hbar c}{Q^2}\right)^{1/2} M_0 \tag{32}$$

where $M_0 = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5}$ g is the Planck mass. Assuming that $Q$ is either the unit electric charge $e$ or the Schwinger magnetic charge $\hbar c/e$, we find $m_0 = 15.9M_0$ for the EMH soliton, or $m_0 = 0.116M_0$ for the EYMH soliton.

Thus, for bona fide test particles (of mass small before the Planck mass), there is no bound state for $l = 0$.

For $l \neq 0$, the preceding argument does not restrict the possible values of $m$. There are bound states for any $l \geqslant 1$, and we may use the inequality (25) to estimate the value of the lowest energy level for a given $l$, in the nonrelativistic approximation $E \simeq m$. In this case

$$\alpha \simeq \frac{m^2\lambda^2 p_0^2}{\hbar^2} \tag{33}$$

from equation (20), and is small for light test particles. Therefore the solution of equation (23) is

$$\eta_0 \simeq \frac{\pi}{2}\left[1 - \frac{\alpha}{l(l+1)}\right] \tag{34}$$

Introducing the nonrelativistic energy $\mathscr{E} \equiv E - m$, we finally obtain from inequality (25) the lower bound

$$\frac{\mathscr{E}}{m} \gtrsim -\frac{1}{2l(l+1)}\left(\frac{GMm}{\hbar c}\right)^2 \{1 - [l(l+1)]^{-1/2}\} \tag{35}$$

which is identical to what we would obtain from a Newtonian potential in the same approximation.

To conclude, the energy levels of a test particle in the field of a gravitational soliton are approximately the same as those of a Newtonian atom, except for the absence of the levels $l = 0$. It should be borne in mind, however, that the binding energies (35) are extremely small, and therefore the level structure is unobservable, unless $M$ and $m$ are not much smaller than the Planck mass.

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